5.1-5.3 nonlinear ODEs

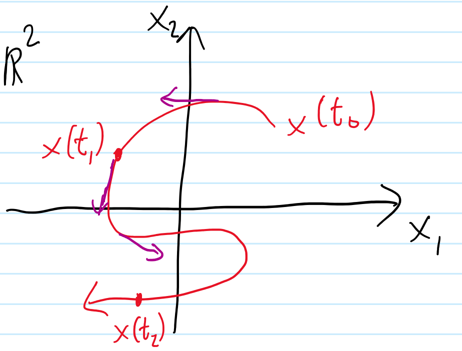
Wednesday, March 17, 2021 3:56 AM

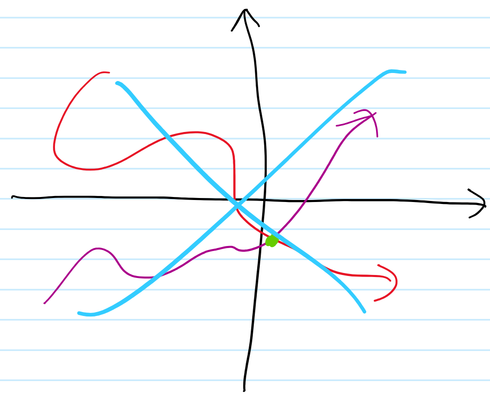
Thm 5.1 Let $F: \mathbb{R}^n \to \mathbb{R}$ (has no explicit t-dependence) be continuous and have continuous partial derivatives $\frac{\partial F}{\partial x_i}$ for $i \in \{1, 2, \dots, n\}$. Then $\exists$ a unique solution to the IVP $\frac{\partial X}{\partial t} = F(X)$ (autonomous system), $X(t_0) = X_0$ for any initial value $X_0 \in \mathbb{R}^n$.

proof. Existence-Uniqueness Thm from DiffEQ.

Define $X(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ parametrically describes a curve in $\mathbb{R}^n$ called the **trajectory (or path or orbit)** of the system and it has a **velocity** $\frac{\partial X}{\partial t}$.

Ex. $\mathbb{R}^2$





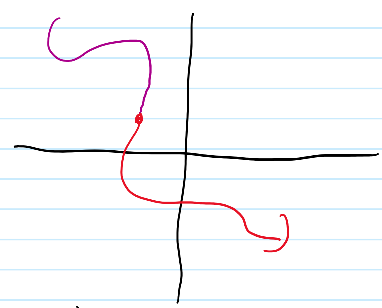
Corollary 5.1 If $F: \mathbb{R}^n \to \mathbb{R}$ is continuous and has n continuous partial derivatives $\frac{\partial F}{\partial x_i}$, then two solutions $X_1(t), X_2(t)$ satisfying $\frac{\partial X}{\partial t} = F(X)$ that have intersecting initial conditions $X_1(0) = X_0$ and $X_2(t_0) = X_0$ must be translations of each other:

$$X_2(t) = X_1(t - t_0)$$

proof. Let $Y(t) = X_1(t - t_0)$ and let $u = t - t_0$.

Then $$\frac{dY(t)}{dt} = \frac{dX_1(u)}{du} \cdot \frac{du}{dt} = \frac{dX_1(u)}{du}$$

$$= F(X_1(u)) = F(X_1(t - t_0)) = F(Y(t)).$$



Additionally, $Y(t_0) = X_1(0) = X_0$, so by existence-uniqueness, $Y(t) = X_2(t)$.

$\boxtimes$

$$Y(t) = X_2(t).$$

Def. 5.1 An equilibrium solution (or steady state, or fixed pt. or critical pt.) of the differential system $\dot{X} = F(X)$ is a constant solution satisfying $F(\bar{X}) = 0$.

Def. 5.2 (viz Def. 2.3) An equilibrium solution is locally stable if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. every sol $X(t)$ of $\dot{X} = F(x)$ with initial condition $X(t_0) = X_0$

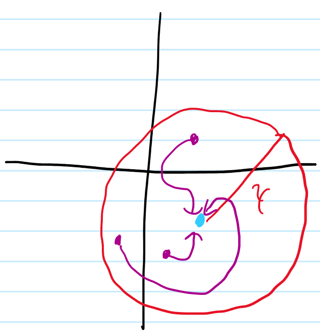
$$\| X_0 - \bar{X} \|_2 < \delta, \text{ satisfies}$$

$$\| X(t) - \bar{X} \|_2 < \varepsilon \qquad \forall t \ge t_0.$$

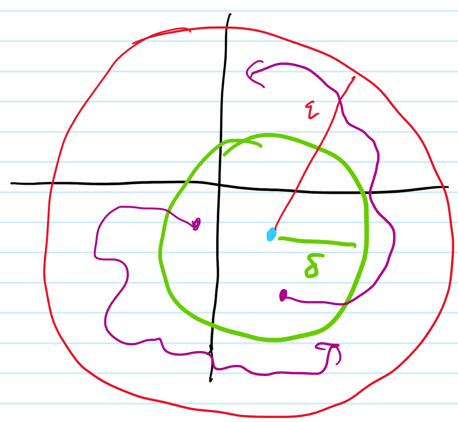
Otherwise, it is unstable.

Def. 5.3 An equilibrium is locally asymptotically stable if it is locally stable and $\exists \gamma > 0$ s.t. $\| X_0 - \bar{X} \| < \gamma$ implies

$$\lim_{t \to \infty} \| X(t) - \bar{X} \|_2 = 0.$$



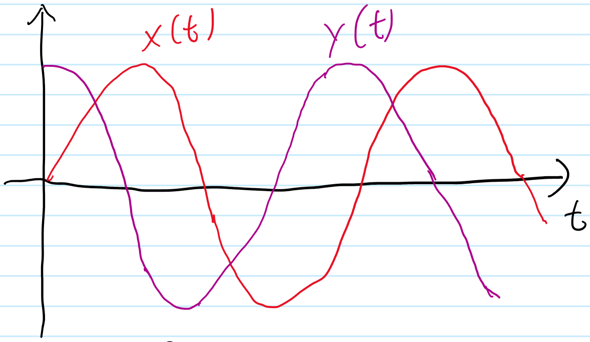
local asympt. stable

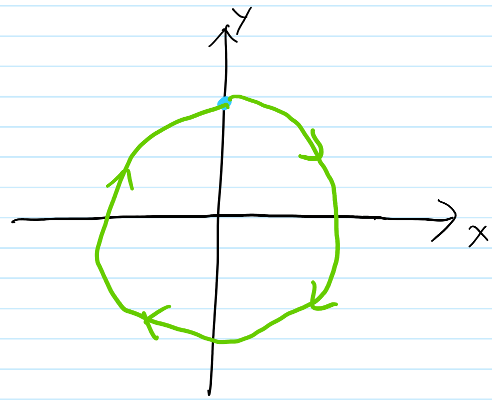


Def. 5.4 A periodic solution is a nonconstant solution satisfying $X(t+T) = X(t)$ for all t on the interval of existence for some $T > 0$. The minimum $T > 0$ is the period.

Ex. $\dfrac{dx}{dt} = y \qquad \dfrac{dy}{dt} = x, \qquad x(0) = 0$ and $y(0) = 1.$

$\Rightarrow \quad x(t) = \sin(t), \quad y(t) = \cos(t)$





Period 2π.

Thm 5.2 Assume $f: \mathbb{R} \to \mathbb{R}$ is continuous. Then $\dot{x} = f(x)$ has no periodic solutions.

proof. Suppose $x(t)$ is a periodic solution with period $T > 0$.

$$\frac{dx(t)}{dt} = f(x)$$

$$\Rightarrow \quad \left(\frac{dx(t)}{dt}\right)^2 = f(x) \cdot \frac{dx(t)}{dt}$$

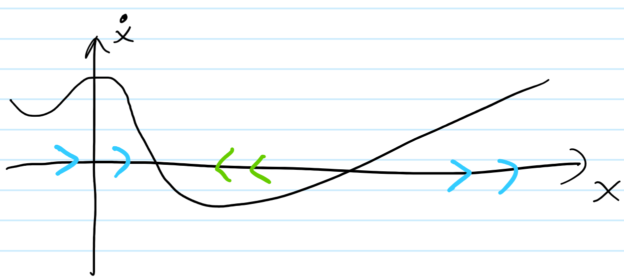
$$\Rightarrow \quad \underbrace{\int_t^{t+T} \left(\frac{dx(s)}{ds}\right)^2 ds}_{>0 \text{ because it is not identically } 0} = \int_t^{t+T} f(x) \cdot \frac{dx(s)}{ds} \cdot ds$$

Let $u = x(s)$

$$= \underbrace{\int_{x(t)}^{x(t+T)} f(u)\, du}_{=0 \text{ by periodicity}}$$

$\Rightarrow$ contradiction, so $x(t)$ can't be a periodic solution. $\boxtimes$

Intuition:



Thm 5.3 Suppose $f': I \to \mathbb{R}$ is continuous and $\bar{x} \in I$, where $\bar{x}$ is an equilibrium of $\dot{x} = f(x)$. Then $\bar{x}$ is locally asymptotically stable if $f'(\bar{x}) < 0$ and unstable if $f'(\bar{x}) > 0$.

Def. 5.5. $\bar{x}$ is **hyperbolic** if $f'(\bar{x}) \neq 0$. $f'(\bar{x})$ is the **eigenvalue** of the linearized equation at $\bar{x}$.